

# The General Stationary Gaussian Markov Process

Larry Brown,<sup>1</sup> Philip Ernst,<sup>1</sup> Larry Shepp<sup>1</sup> and Robert Wolpert<sup>2</sup>

<sup>1</sup>Department of Statistics,  
The Wharton School of the University of Pennsylvania

<sup>2</sup>Department of Statistical Science, Duke University

January 3, 2014

## Abstract

We find the class,  $\mathcal{C}_k, k \geq 0$ , of all zero mean stationary Gaussian processes,  $Y(t), t \in \mathbb{R}$  with  $k$  derivatives, for which

$$Z(t) \equiv (Y^{(0)}(t), Y^{(1)}(t), \dots, Y^{(k)}(t)), t \geq 0$$

is a  $(k+1)$ -vector Markov process. (here,  $Y^{(0)}(t) = Y(t)$ ).

## 1 Introduction

We show that the process,  $Y$ , can be described in three equivalent ways:

(i). Each member of  $\mathcal{C}_k$  is given uniquely by a certain polynomial  $P(z)$  via the covariance of any such  $Y$ :

$$\mathbb{E}[Y(s)Y(t)] = R(s, t) = r(t - s) = \int_{-\infty}^{\infty} \frac{e^{i(t-s)z} dz}{|P(z)|^2}$$

where  $P(z)$  is a polynomial of degree  $k+1$  with positive leading coefficient and all complex roots  $\zeta_j = \rho_j + i\sigma_j, j = 0, \dots, k$  with  $\sigma_j > 0$ , and  $\rho_j$  real. If some  $\rho_j \neq 0$ , then there is another  $\zeta_k = -\zeta_j^*$  which is the negative conjugate of  $\zeta_j$ . Note  $r(t)$  automatically has  $2k$  derivatives at each  $t$ , but not  $2k+1$ .

(ii) Equivalently, it is necessary and sufficient that  $Y \in \mathcal{C}$  has the representations via Wiener integrals with a standard Brownian motions,  $W$

$$Y(t) = \int_{-\infty}^{\infty} g(t - \theta) dW(\theta)$$

where  $g$  has  $L^2$  Fourier transform,

$$\hat{g}(z) = \frac{1}{|P(z)|}$$

or, equivalently,  $Y$  has the spectral representation, with a pair of independent standard Brownian motions,  $W_1, W_2$ , with  $f(z) \equiv \hat{g}$ ,

$$Y(t) = \int_{-\infty}^{\infty} \cos tz f(z) dW_1(z) + \int_{-\infty}^{\infty} \sin tz f(z) dW_2(z)$$

(iii) Equivalently, it is necessary and sufficient for  $Y \in \mathcal{C}$  that  $Z$  has a representation as an Ito vector diffusion process:

$$\begin{aligned} dY^{(i)}(t) &= Y^{(i+1)}(t)dt, \quad 0 \leq i < k, t \in \mathbb{R} \\ dY^{(k)}(t) &= \sum_{j=0}^k a_j Y^{(j)}(t)dt + b dW(t), \quad t \in \mathbb{R} \end{aligned}$$

where the coefficients,  $a_j$ , are the unique solution of the equations:

$$r^{(k+i+1)}(0^+) = \sum_{j=0}^k a_j r^{(i+j)}(0), \quad i = 0, 1, \dots, k$$

Note that the left and right derivatives of  $r^{(j)}$  are equal except for  $j = 2k$ . The diffusion coefficient,  $b$ , is given by

$$b^2 = \sum_{j=0}^k a_j r^{(j+k)}(0)(-1)^{j+1} + (-1)^k r^{(2k+1)}(0^-)$$

Given the polynomial,  $P(z)$ , in (i), which uniquely determines each process  $Y \in \mathcal{C}$ , the derivatives  $r^{(j)}(0)$  are easily obtained from the representation in (i) above as

$$r^{(j)}(0) = \int_{-\infty}^{\infty} \frac{(iz)^j dz}{|P(z)|^2}, \quad 0 \leq j \leq 2k+1$$

where for  $j = 2k+1$  the integral is not  $L^1$  convergent, but is understood as a principal value. Then the coefficients  $a_j, b$  of the Ito equation is determined as indicated above. This allows one to determine exactly which Ito vector equations have stationary solutions, and what the stationary distribution is. Of course it is the Gaussian vector,  $Y^{(j)}(0)$ , with covariance

$$r^{(i+j)}(0), \quad i, j = 0, \dots, k$$

It seems that these results are new despite the fact that the problem has been around for nearly fifty years to describe the class  $\mathcal{C}$ .

## 2 The case $k = 0$

For  $k = 0$ , we ask what is the set of all stationary Gauss-Markov processes and the answer is well-known as the set of processes with covariance  $r(t) = Ae^{-\alpha|t|}$ ,  $A \geq 0$ ,  $\alpha \geq 0$ , with representation

$$Y(t) = \sqrt{A}e^{-\alpha t}W(e^{2\alpha t})$$

with  $W$  a standard Wiener process, which satisfies an Ito equation of the form

$$dY(t) = aY(t)dt + b dW(t), \text{ where } a < 0 \text{ and } b > 0$$

Let us give the simple proof for  $k = 0$  that the covariance  $r$  must be as stated, because we will use the same method for the general case,  $k$ , and this will make things clearer. The idea is to find or define the covariance,  $R(s, t) = r(|t - s|)$ , under the stated assumptions. We must have  $Y(s)$  and  $Y(u)$  given  $Y(t)$  to be uncorrelated for Markovianness to hold, whenever  $s < t < u$ . Since

$$\mathbb{E}[Y(u) \mid Y(t)] = Y(t) \frac{r(t - s)}{r(0)}$$

this means that  $r$  satisfies for any positive  $u, v$ ,

$$\mathbb{E} \left[ Y(u) - \frac{r(u)}{r(0)} Y(0) \right] Y(-v) = 0, \text{ or } r(u + v) = \frac{r(u)r(v)}{r(0)}$$

Since  $r$  is continuous and nonnegative definite, it follows that  $r(h) = Ae^{-h\alpha}$ ,  $h \geq 0$ . To see this note that  $f(t) = \log \frac{r(t)}{r(0)}$  satisfies  $f(u + v) = f(u) + f(v)$  and so  $f(\frac{m}{n}) = f(1)\frac{m}{n}$ . Since  $f$  is continuous we have  $f(u) = uf(1)$  and the claim follows. Since  $r$  is to be nonnegative definite we must have  $A \geq 0$  and  $\alpha \geq 0$ . We have found a necessary condition for  $r$  to be the covariance. In fact we see that  $r$  is infinitely differentiable, except at  $u = 0$ , where  $r$  has finite left and right derivatives. An analogous property will hold for every  $k$ . The only thing missing is to prove sufficiency, i.e., the existence of a process with this covariance. This is easy if we use the representation:

$$Y(t) = \sqrt{A}e^{-t\alpha}W(e^{2\alpha t})$$

which has covariance  $r(t - s) = Ae^{-|t-s|\alpha}$ .

We know  $Y$  is Markovian, so we can obtain the coefficients of the Ito equation by the formula,

$$aY(0) = \lim_{h \downarrow 0} \frac{\mathbb{E}[Y(h) - Y(0) \mid Y(0)]}{h} = \lim_{h \downarrow 0} \frac{Y(0)(r(h) - r(0))}{hr(0)} = -\alpha Y(0)$$

and the other Ito coefficient is

$$b = \lim_{h \downarrow 0} \frac{(\mathbb{E}[Y(h)] - \mathbb{E}[Y(h) | Y(0)])^2}{h} = \lim_{h \downarrow 0} \frac{r(0) - \frac{r^2(h)}{r(0)}}{h} = 2\alpha A$$

so the Ito equation is

$$dY(t) = -\alpha Y(t)dt + \sqrt{2A\alpha}dW(t)$$

Finally, the stationary measure has  $\sigma^2 = r(0) = A$ .

### 3 The case $k = 1$

We will give the approach for general  $k$  but let's do  $k = 1$ . Since  $Y(u)$  and  $Y(s)$  are conditionally uncorrelated given  $Y(t)$  for  $s < t < u$ , we need

$$(\mathbb{E}[Y(u)] - \mathbb{E}[Y(u) | Y(0), Y'(0)])Y(-v) = 0,$$

for  $u > 0 > -v$ , and since we must have  $r'(0) = 0$  since  $r$  is even and is differentiable at zero because  $Y$  is differentiable, we have

$$\mathbb{E}[Y(u) | Y(0), Y'(0)] = \frac{r(u)}{r(0)}Y(0) + \frac{r'(u)}{r''(0)}Y'(0)$$

Since  $(\mathbb{E}[Y(u)] - \mathbb{E}[Y(u) | Y(0), Y'(0)])Y(-v) = 0$ , for  $u > 0, v > 0$ , this gives

$$(*) \quad r(u+v) - \frac{r(u)r(v)}{r(0)} - \frac{r'(u)r'(v)}{r''(0)} = 0, u > 0, v > 0$$

This shows that  $r(u), u > 0$  is infinitely differentiable because  $r$  is differentiable and  $(*)$  exhibits  $r'$  in terms of  $r$ , so that  $r'$  is differentiable, and by induction  $r$  has all derivatives at  $u \neq 0$ . Moreover, since  $r$  is even we must

have  $r'(0) = 0$ . We next show that  $r(u)$  satisfies a second order ODE with constant coefficients, namely:

$$r^{(2)}(u) = r^{(0)}(u) \frac{r^{(2)}(0)}{r^{(0)}(0)} + r^{(1)}(u) \frac{r^{(3)}(0+)}{r^{(2)}(0)}$$

We claim first that  $r$  is twice differentiable at  $u = 0$ . This follows from the fact that  $Y'(t)$  is a Gaussian variable and so

$$r''(0) = -\mathbb{E}[Y'(0)Y'(0)]$$

exists.

If we expand each side of (\*) into power series in  $v$ , then we get that up to a term,  $o(v^2)$ ,

$$\sum_{j=0}^2 \frac{r^{(j)}(u)v^j}{j!} = \frac{r(u)}{r(0)} \sum_{j=0}^2 \frac{r^{(j)}(0)v^j}{j!} + \frac{r^{(1)}(u)}{r^{(2)}(0)} \sum_{j=0}^2 \frac{r^{(j+1)}(0)v^j}{j!}$$

and we see that the coefficients of  $v^0, v^1$  vanish automatically, but the coefficient of  $v^2$  shows that  $r^{(2)}(0+)$  exists and then the coefficient  $v^2$  being equal on both sides gives a second degree differential equation for  $r$ , for  $u > 0$ . Thinking of  $r(0), r^{(2)}(0), r^{(3)}(0+)$  as constants, we see that for  $u > 0$ , the differential equation for  $r$  has constant coefficients. If the indicial equation has distinct roots, then this means that

$$r(u) = \sum_{j=1}^2 A_j e^{-ua_j}. \quad (1)$$

If the two roots are not distinct, then one gets a limiting covariance e.g., for the case when  $a_1 = a, a_2 = a + \epsilon$ , and  $A_1 = -A_2 = \frac{1}{\epsilon}$ ,  $r$  becomes the derivative,

$$r(u) = (1 + au)e^{-ua}, u \geq 0$$

Next, we have to check that any such  $r$  satisfies equation (\*). This is easy to check in this case, so that satisfying (\*) imposes no additional restrictions

than satisfying a second order ode with constant coefficients. It is not true that every such  $r$  is realizable because  $r$  must be a covariance and conditions to ensure this must be placed on  $a_j$  and  $A_j$ . For example we must have  $a_j > 0$ . For  $Y$  to be differentiable we need that  $r(h)$  be twice differentiable at  $h = 0$ . In turn, this means that  $-r'(0) = A_1 a_1 + A_2 a_2 = 0$ . We also need that  $r(0) = A_1 + A_2 > 0$ . Let us use the process represented below to show that with these restrictions the condition is also sufficient to realize the covariance in (1):

$$Y(t) = \int_{-\infty}^{\infty} f(t - \theta) dW(\theta)$$

where  $f$  is any  $L^2$  function to get a class of processes with covariance  $r$  of the form above. Set:

$$f(x) = A^- e^{x a^-}, \quad x < 0, \quad f(x) = A^+ e^{-x a^+}, \quad x > 0$$

Now,  $Y$  will only be well defined when  $f \in L^2$ , so we need  $a^\pm > 0$ . The covariance of the representation is easily seen to be

$$r(u) = \left( \frac{(A^-)^2}{2a^-} - \frac{A^- A^+}{a^- - a^+} \right) e^{-h a^-} + \left( \frac{(A^+)^2}{2a^+} + \frac{A^- A^+}{a^- - a^+} \right) e^{-h a^+}$$

Also  $Y$  will only be differentiable when  $f$  is continuous. so this means  $f(0-) = f(0+)$  or  $A^- = A^+$ . We get a certain class of covariances of our form. Wolog we can choose  $a_1 = a^-, a_2 = a^+$ . If we set  $A^+ = A^- = A$ , we need to choose  $A$  so that

$$A_1 = A^2 \left( \frac{1}{2a_-} - \frac{1}{a_- - a_+} \right), \quad A_2 = A^2 \left( \frac{1}{2a_+} + \frac{1}{a_- - a_+} \right)$$

It is easy to check that  $a_1 A_1 + a_2 A_2 = 0$  holds.

### 3.1 Discrete Considerations

Using definition (iii) to classify the elements  $\mathcal{C}_k$ , we ask for the form of an AR(2) process that will give rise to the continuous AR(2) process.

## 4 General $k \geq 2$

We must have  $Y(s)$  and  $Y(u)$  conditionally independent given  $Y^{(j)}(t), j = 0, \dots, k$ , and since the process is Gaussian this means that  $Y(s)$  and  $Y(t)$  are conditionally uncorrelated. This means that

$$\mathbb{E} \left[ Y(u) - \sum_{j=0}^k \alpha_j(u) Y^{(j)}(0) \right] Y(-v) = 0$$

for  $u > 0, v > 0$  where

$$\mathbb{E} [Y(u) \mid Y^{(j)}(0), j = 0, \dots, k] = \sum_{j=0}^k \alpha_j(u) Y^{(j)}(0)$$

because conditional expectations are linear for Gaussian processes. Note the  $\alpha_j$ 's are defined uniquely by the equations

$$r^{(i)}(u) = \sum_{j=0}^k \alpha_j(u) r^{(i+j)}(0), \quad i = 0, \dots, k.$$

We would like to show that, without any further assumptions than the fact that  $r$  satisfies an ode of degree  $k+1$ , that the first equation holds, because then we can conclude that the first equation poses no additional restrictions. The first equation is the same as

$$r(u+v) = \sum_{j=0}^k \alpha_j(u) r^{(j)}(v), \quad u > 0, \quad v > 0$$

i.e.,  $r(u+v)$  is of rank  $k+1$ , i.e., (2) holds if the  $\alpha_j$ 's are defined by (2). Here is where using the first approach pays off to avoid a lot of algebra. Imagine solving (2) for the  $\alpha_j(u)$ 's and then placing these  $\alpha_j(u)$  into (2). We now let  $v$  be small and positive and use power series. We get that  $r(u)$  satisfies a



differential equation of degree  $k + 1$  with constant coefficients as in the cases  $k = 0, 1$ . But to avoid checking that no further condition is required to prove that  $r$  also satisfies (2) we can argue as follows. Let for  $v$  fixed,

$$f(u) = r(u + v) - \sum_{j=0}^k \alpha_j(u) r^{(j)}(v), \quad u > 0$$

Note that  $r(u + v)$  and  $r^{(j)}(u)$ , as functions of  $u$ , for all  $j$  and any fixed  $v$  satisfy the same differential equation because the differential equation has constant coefficients. Also, there are  $k + 1$  zero boundary conditions  $f^{(j)}(0+) = 0$ , so that  $f \equiv 0$ . This is quite subtle, and we need that  $u > 0$  and  $v > 0$  here to get the required differentiability. It follows that (2) holds.

So we have proved that if  $r$  is a covariance for which  $Z$  is a  $(k + 1)$ -vector Markov process, then  $r(u), u > 0$  satisfies a differential equation of degree  $k + 1$  with constant coefficients. The general solution for  $r(u)$  must also be of the form (since  $r$  is nonnegative definite):

$$r(u) = \int_{\mathbb{R}} e^{iux} \mu(dx)$$

for some nonnegative (spectral) measure,  $\mu$ . Since  $r(u) = r(-u)$ ,  $\mu$  is even, and since  $r$  satisfies a differential equation of order  $k + 1$  with constant coefficients, we must have for  $u \neq 0$ ,

$$\sum_{j=0}^k b_j r^{(j)}(u) = 0 = \int_{\mathbb{R}} \sum_{j=0}^k b_j (-ix)^j e^{iux} \mu(dx) = \int_{\mathbb{R}} e^{ixu} P(x) \mu(dx)$$

We next prove that we must have, with  $c > 0$ ,

$$r(t) = \int_{\mathbb{R}} \frac{e^{izt}}{|P(z)|^2} dz$$

where  $P(z) = c \prod_{j=0}^k (1 - \frac{z}{\zeta_j})$ , with  $c > 0$ , and with the  $k + 1$  complex numbers,  $\zeta_j, j = 0, \dots, k$  having strictly positive imaginary part. We require that  $f(z) = f(-z)$  so we must have for each  $\zeta_j$  another  $\zeta_{j'}$  for which  $\zeta_j = -\zeta_{j'}^*$

is the negative complex conjugate. It may be that  $j' = j$  in which case  $\zeta_j$  is on the positive imaginary axis. For such a polynomial  $P$ , there is an ode with constant coefficients satisfied by  $r(t), t > 0$  because, by Cauchy's theorem, the differential operator  $P(-iD)r(t)$ ,  $D = \frac{d}{dt}$ , is just

$$P(-iD)r(t) = \int_{\mathbb{R}} \frac{P(z)}{P(z)P^*(z)} e^{izt} dz = \int_{\mathbb{R}} \frac{e^{itz}}{P^*(z)} dz = 0$$

because we can complete the integral by adding a semicircle above the real  $z$ -axis along which, for  $u > 0$ ,  $e^{iuz}$  is bounded, and since the last integrand is analytic in the upper half plane the integral is zero, and as the radius of the semicircle goes to infinity the contribution from the arc is negligible because  $P^*(z)$  is large.

The representation of  $r$  as a covariance is now immediate, since we can just set

$$Y(t, \omega) = \int_{\mathbb{R}} \cos tz \frac{dW_1(z, \omega)}{|P(z)|} + \int_{\mathbb{R}} \sin tz \frac{dW_2(z, \omega)}{|P(z)|}$$

where  $W_i$  are iid standard Brownian motions, and check that  $Y$  has covariance  $r$ . Note that we cannot have any polynomial factor in the numerator of the above equation for  $r$  because  $r^{(2k)}(u)$  must exist.

The Ito equation for

$$Z(t) = (Y^{(0)}(t), Y^{(1)}(t), \dots, Y^{(k)}(t))$$

is degenerate for the coefficients of  $dY^{(j)}(t), j < k$ , since

$$dY^{(i)}(t) \equiv Y^{(i+1)}(t)dt, i < k$$

but for  $i = k$ , we need to compute  $a_j$  and  $b$  in

$$dY^{(k)}(t) = \sum_{j=0}^k a_j Y^{(j)}(t)dt + b dW(t)$$

The  $a_j$ 's are found as follows, we may as well take  $t = 0$ , so

$$\begin{aligned}
\sum_{j=0}^k a_j Y^{(j)}(0) &= \lim_{h \downarrow 0} \frac{\mathbb{E} [Y^{(k)}(h) - Y^{(k)}(0) \mid Y^{(0)}(0), \dots, Y^{(k)}(0)]}{h} \\
&= \lim_{h \downarrow 0} \frac{\sum_{j=0}^k \alpha_j^{(k)}(h) Y^{(j)}(0) - Y^{(k)}(0)}{h} = \sum_{j=0}^k \alpha_j^{(k+1)}(0) Y^{(j)}(0)
\end{aligned}$$

where we have used the fact that  $\alpha_j^{(i)}(0) = \delta_{i,j}$  because  $\alpha_j^{(i)}(h)$  satisfies, for any  $i \geq 0$ ,

$$r^{(i)}(h) = \sum_{j=0}^k \alpha_j^{(i)}(h) r^{(j)}(0)$$

and we may set  $h = 0$ . Comparing coefficients, we can read off the result,  
 $a_j = \alpha_j^{(k+1)}(0)$ .

Moreover, the values of  $a_j$  are given as the solutions of the equations satisfied by the  $\alpha_j^{(i)}$ 's, i.e.,

$$r^{(i+k+1)}(0) = \sum_{j=0}^k a_j r^{(i+j)}(0), i = 0, 1, \dots, k$$

so the values of  $a_j$  are now determined. To determine  $b$ , we use

$$\begin{aligned}
b^2 &= \lim_{h \downarrow 0} \frac{(\mathbb{E} [Y^{(k)}(h)] - \mathbb{E} [Y^{(k)}(h) \mid Y^{(0)}, \dots, Y^{(k)}(0)])^2}{h} \\
&= \lim_{h \downarrow 0} \frac{r^{(2k)}(0)(-1)^k - \sum_{j=0}^k \alpha_j^{(k)}(h) r^{(k+j)}(h)(-1)^j}{h} \\
&= \sum_{j=0}^{k-1} \alpha_j^{(k+1)}(0) r^{(k+j)}(0)(-1)^{j+1} + (-1)^{k+1} \lim_{h \downarrow 0} \frac{r^{(2k)}(h) \alpha_k^{(k)}(h) - r^{(2k)}(0) \alpha_k^{(k)}(0)}{h}.
\end{aligned}$$

but the last limit is just the value of the derivative of the product of  $r^{(2k)}(h) \alpha_k^{(k)}(h)$  at  $h = 0$ , so we finally arrive at

$$b^2 = \sum_{j=0}^k \alpha_j^{(k+1)}(0) r^{(k+j)}(0) (-1)^{j+1} + (-1)^{k+1} r^{(2k+1)}(0^+)$$

or in terms of the already calculated  $a_j$ 's, switching to  $0^-$

$$b^2 = \sum_{j=0}^k a_j r^{(k+j)}(0) (-1)^{j+1} + (-1)^k r^{(2k+1)}(0^-)$$

All the coefficients can be computed in terms of the unique polynomial  $P$  which corresponds to any process  $Y$  in  $\mathcal{C}$  because the only thing we need to know are the derivatives of  $r(t)$  at  $t = 0$ , which are given by

$$r^{(j)}(0^\pm) = \mp \int_{\mathbb{R}} \frac{(iz)^j dz}{|P(z)|^2}$$

where the integral is absolutely convergent for  $j \leq 2k$  and is understood as a principle value integral for  $j = 2k + 1$ .

### Generalizations

The problem also makes sense for  $k = \infty$ : the paths of  $Y$  are then entire analytic functions. In the case  $k = \infty$ , the Markov process degenerates because the values of  $Y^{(j)}(0)$  for all  $j$  completely determines the past and the future of  $Y$  because of the power series representation,

$$Y(t) = \sum_{j=0}^{\infty} \frac{Y^{(j)}(0) t^j}{j!}$$

Another generalization is to allow  $Y$  itself to be a vector process,

$$\mathbf{Y}(t) = (Y_1(t), \dots, Y_n(t))$$

and ask the same question. When is  $\mathbf{Y}$  together with its first  $k$  derivatives a mean zero stationary Gaussian Markov process. It seems there is no real

trouble making this generalization, although noncommuting matrices enter. Indeed, the equation for the covariance,

$$\mathbf{R}(s, t) = \mathbb{E}[\mathbf{Y}(s)\mathbf{Y}(t)] = \mathbf{r}(t - s)$$

even when  $k = 0$  is

$$r_{ij}(u + v) = \sum_{k,l} r_{ik}(u) A_{kl} r_{lj}(v)$$

for some matrix  $A$  which may not commute with the matrix  $r_{ij}(u)$ .